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THREE-DIMENSIONAL LARGE DEFORMATION ANALYSIS OF THIN WALLED BEAMS

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Abstract—A set of kinematic assumptions and constraints is used to develop a formulation for the analysis of thin-walled beams, of arbitrary open cross-section, subjected to arbitrarily large displacements in three-dimensional space. The validity of the formulation is illustrated through numerical solutions for elastic lateral-torsional post-buckling behavior of "I" beams, and a comparison of these solutions with experimental results.

NOTATION

$B^{\alpha\beta}$	stress resultant (eqn 29)
$C_\beta{}^\alpha$	coefficients defined in eqn (24)
e_{ij}	three-dimensional strain tensor
$\mathbf{e}_0$	base vector associated with s
$\mathbf{e}_\alpha$	base vector associated with s^α
$\mathbf{f}_\alpha$	unit vector along center-line of element α
$g_{00}, g_{0\alpha}, g_{\alpha\beta}$	metric measures introduced in eqns (5)–(7)
L	length of beam axis
M^α	stress resultant, eqn (28)
N	stress resultant, eqn (27)
n	number of elements in the cross section
$\mathbf{n}_\alpha$	unit vector normal to $\mathbf{f}_\alpha$ in the plane of the cross-section
$Q^{\alpha\beta}$	stress resultant, eqn (31)
$\mathbf{r}$	position vector
S^α	stress resultant, eqn (30)
s	coordinate along the axis of the beam
s^α	cross-section coordinates
T^α	see eqn (22)
$\hat{t}^{ij}$	stress tensor
$\mathbf{v}$	displacement vector of a point on the axis of the beam
$\bar{\epsilon}_{00}, \bar{\epsilon}_{0\alpha}, \bar{\epsilon}_{\alpha\beta}$	mean strain measures, eqns (14)–(16)
$\phi_1, \phi_2, \phi_3, \theta_1, \theta_2, \theta_3$	polar parameters used in numerical examples, eqns (38), (39)
σ_{ij}	Kirchhoff's stress tensor
ω_α	change in specific twist in element α
Ω	volume
$\hat{}$	symbol to denote a magnitude after deformation
$'$	differentiation with respect to s.

1. INTRODUCTION

Thin walled beams are the most often used elements in steel construction. The small deformation theory of such elements as developed by Vlasov[1] is well known and has found its way into everyday engineering practice. Yet, to the authors' knowledge, a general theory to cover the case of large three-dimensional deformations does not exist. The importance of such a theory from the practical standpoint is mainly the prediction of post-buckling behaviour in the elastic and plastic ranges. But there are some aspects in the theory interesting by themselves. The nonlinear theory of rods has achieved in recent years a great degree of generality. The use of projection methods[2, 3] and of the concept of a generalized one-dimensional continuum[4, 5] are the two main approaches to the subject. However, in the first case, the specialization of the results to a thin walled beam is not straightforward and, in the second case, the use of only two directors in the plane of the beam cross-section, as is generally done, is not enough for a representation of the typical behaviour of thin walled beams.

The purpose of this paper is to present a large deformation theory for thin walled beams. The method used, based on integration of stress components, is close to a projection approach but perhaps less rigorous. The results could be reinterpreted, however, from the viewpoint of a generalized continuum. This reinterpretation is not attempted in the present work.

Keeping in mind the practical application of the theory, a numerical formulation for the case of initially straight beams of constant "I" cross-section is outlined. The discretization of the field equations is achieved by the method of finite elements. Numerical examples are presented to illustrate the application of the theory.

2. CROSS-SECTION GEOMETRY

The (constant) cross-section may be composed of any number n of straight thin rectangular elements, as shown in Fig. 1. It is assumed that the cross-section is simply connected, i.e. closed branches are ruled out. A unit vector $\mathbf{f}_\alpha$ is defined along the centerline of each element α ($\alpha = 1, 2, \ldots, n$). At each cross-section an arbitrary origin "O" is selected on the centerline of one of its elements in such a way that the set of all "O"-s forms a smooth space curve, called the axis of the beam, which may be expressed by the equation

$$\mathbf{r}_0 = \mathbf{r}_0(s) \qquad 0 \le s \le L \tag{1}$$

in which s measures length along this space curve and L is the total length of the axis.

A set of length-measuring coordinates s^α is defined in the cross-section, with the following properties: (a) s^α increases in the direction of $\mathbf{f}_\alpha$; (b) the first point of element α at which one arrives in tracing the (unique) simple trajectory (along element centerlines) starting at "O", is called the origin of element α. At that origin $s^\alpha = 0$; (c) for $\beta \neq \alpha$, on element α, $s^\beta \equiv$ the last value attained by s^β in the trajectory; or zero, if element β is not touched by the trajectory in reaching element α.

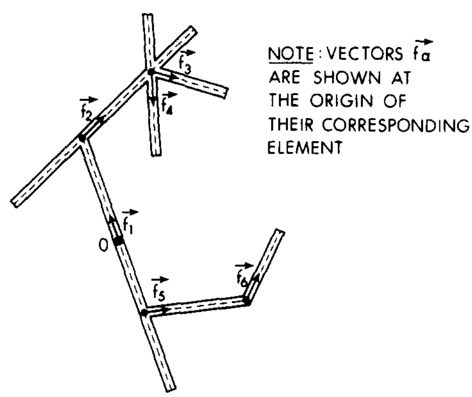

Fig. 1. Cross-section geometry.

3. KINEMATIC GENERALITIES

Throughout this section the vectors $\mathbf{f}_\alpha$ will be assumed to be material and to transform, after deformation, into another set of vectors $\hat{\mathbf{f}}_\alpha$. Under this assumption a thickness-wise mean strain tensor will be constructed. The deviations from the mean will then be accounted for by additional assumptions.

According to the definitions of the preceding section, the initial position vector $\mathbf{r}$ to a point on the center line of any element in the cross-section corresponding to $\mathbf{r}_0$, is given by:

$$\mathbf{r} = \mathbf{r}_0 + s^\alpha \mathbf{f}_\alpha \tag{2}$$

in which, as in what follows, the summation convention applies to any index appearing in a product once as a superscript and once as a subscript. The range of the summation should be clear from the context.

Differentiating eqn (2) with respect to s and with respect to s^α the following results are obtained:

$$\mathbf{e}_0 = \frac{\partial \mathbf{r}}{\partial s} = \mathbf{r}_0' + s^\alpha \mathbf{f}_\alpha' \tag{3}$$

$$\mathbf{e}_\alpha = \frac{\partial \mathbf{r}}{\partial s^\alpha} = \mathbf{f}_\alpha \qquad \alpha = 1, 2, \ldots, n \tag{4}$$

in which $(\ )' = (\partial/\partial s)(\ )$.

Note that the meaning of $\mathbf{e}_0$ and $\mathbf{e}_\alpha$ is not that $n+1$ different base vectors are present at each point, but rather that in a two-dimensional manifold the first base vector, $\mathbf{e}_0$, varies from point to point while the second base vector, $\mathbf{e}_\alpha$, varies from element to element as identified by α.

Forming all the possible scalar products of eqns (3) and (4) yields:

$$g_{00} = \mathbf{r}_0' \cdot \mathbf{r}_0' + 2s^\alpha \mathbf{r}_0' \cdot \mathbf{f}_\alpha' + s^\alpha s^\beta \mathbf{f}_\alpha' \cdot \mathbf{f}_\beta' \tag{5}$$

$$g_{0\alpha} = \mathbf{r}_0' \cdot \mathbf{f}_\alpha + s^\beta \mathbf{f}_\beta' \cdot \mathbf{f}_\alpha \tag{6}$$

$$g_{\alpha\beta} = \mathbf{f}_\alpha \cdot \mathbf{f}_\beta. \tag{7}$$

Note, again, that g_{ij} $(i = 0, \ldots, n)$ is not a metric tensor of an $(n+1)$-dimensional manifold, because of the interpretation described above.

Using the symbol ˆ to denote a magnitude after deformation and denoting by $\mathbf{v}$ the displacement vector of a point on the axis of the beam, the position vector after deformation may be written as

$$\hat{\mathbf{r}} = \mathbf{r}_0 + \mathbf{v} + s^\alpha \hat{\mathbf{f}}_\alpha. \tag{8}$$

Consideration of s as a convected coordinate yields the following formulae for the base vectors and their scalar products after deformation:

$$\hat{\mathbf{e}}_0 = \mathbf{r}_0' + \mathbf{v}' + s^\alpha \hat{\mathbf{f}}_\alpha' \tag{9}$$

$$\hat{\mathbf{e}}_\alpha = \hat{\mathbf{f}}_\alpha \tag{10}$$

$$\hat{g}_{00} = \mathbf{r}_0' \cdot \mathbf{r}_0' + 2\mathbf{r}_0' \cdot \mathbf{v}' + 2s^\alpha \mathbf{r}_0' \cdot \hat{\mathbf{f}}_\alpha + 2\mathbf{v}' \cdot s^\alpha \hat{\mathbf{f}}_\alpha' + \mathbf{v}' \cdot \mathbf{v}' + s^\alpha s^\beta \hat{\mathbf{f}}_\alpha' \cdot \hat{\mathbf{f}}_\beta' \tag{11}$$

$$\hat{g}_{0\alpha} = \mathbf{r}_0' \cdot \hat{\mathbf{f}}_\alpha + \mathbf{v}' \cdot \hat{\mathbf{f}}_\alpha + s^\beta \hat{\mathbf{f}}_\beta' \cdot \hat{\mathbf{f}}_\alpha \tag{12}$$

$$\hat{g}_{\alpha\beta} = \hat{\mathbf{f}}_\alpha \cdot \hat{\mathbf{f}}_\beta. \tag{13}$$

By subtracting eqns (5)–(7) from eqns (11)–(13), respectively, and dividing by two, the following Green strain measures $\bar{\epsilon}_{ij}$ are obtained:

$$\bar{\epsilon}_{00} = \mathbf{r}_0' \cdot [\mathbf{v}' + s^\alpha(\hat{\mathbf{f}}_\alpha' - \mathbf{f}_\alpha')] + \mathbf{v}' \cdot \left[\frac{1}{2}\mathbf{v}' + s^\alpha \hat{\mathbf{f}}_\alpha'\right] + \frac{1}{2} s^\alpha s^\beta [\hat{\mathbf{f}}_\alpha' \cdot \hat{\mathbf{f}}_\beta' - \mathbf{f}_\alpha' \cdot \mathbf{f}_\beta'] \tag{14}$$

$$\bar{\epsilon}_{0\alpha} = \frac{1}{2}\mathbf{r}_0' \cdot [\hat{\mathbf{f}}_\alpha - \mathbf{f}_\alpha] + \frac{1}{2}\mathbf{v}' \cdot \hat{\mathbf{f}}_\alpha + \frac{1}{2} s^\beta [\hat{\mathbf{f}}_\beta' \cdot \hat{\mathbf{f}}_\alpha - \mathbf{f}_\beta' \cdot \mathbf{f}_\alpha] \tag{15}$$

$$\bar{\epsilon}_{\alpha\beta} = \frac{1}{2}[\hat{\mathbf{f}}_\alpha \cdot \hat{\mathbf{f}}_\beta - \mathbf{f}_\alpha \cdot \mathbf{f}_\beta]. \tag{16}$$

Note that although $\bar{\epsilon}_{0\alpha}$ is defined for all points, its interpretation as a thickness-wise mean shear strain meterializes only when s^β is evaluated for element α.

4. CONSTRAINTS

So far, only a general kinematic description has been presented. In order to construct a meaningful theory, however, that description has to be supplemented with a certain number of constraints limiting the possible deformed configurations.

The first group of constraints will reflect the usually adopted hypothesis that transverse normal strains are negligible. These constraints are expressed by

$$\bar{\epsilon}_{\alpha\beta} = 0 \qquad \text{for} \qquad \alpha = \beta. \tag{17}$$

The second group of constraints to be introduced is motivated by the need of avoiding rigid body rotations of one element with respect to the others. This might occur in a straight "I" beam, for instance, by one of the flanges rotating rigidly around its juncture with the web. These constraints are satisfied after assuming the validity of eqn (17) by the condition

$$\bar{\epsilon}_{\alpha\beta} = 0 \qquad \text{for } \alpha, \beta \text{ contiguous elements.} \tag{18}$$

Further constraints might be adopted as, for instance, the vanishing of mean shear strains. Thus, in the linear theory of Vlasov[1] the vanishing of those strains yields, by integration, an explicit expression of one of the displacement components in terms of the other. In the nonlinear theory this explicit integration is not viable and the constraint becomes cumbersome. The theory developed here will account, therefore, for a constant mean shear strain in each element. (Note that the constancy of $\bar{\epsilon}_{0\alpha}$ in element α stems from eqns (17)–(15)).

5. INTERNAL VIRTUAL WORK EXPRESSION

The expression for the internal virtual work (IVW) is:

$$\text{IVW} = \int_{\hat{\Omega}} \hat{t}^{ij} \delta e_{ij} \, d\hat{\Omega} \qquad i,j = 1, 2, 3 \tag{19}$$

in which $\hat{t}_{ij}$ = Cauchy's stress tensor, e_{ij} = the three-dimensional Green's strain tensor, and $\hat{\Omega}$ = volume in the deformed configuration. In terms of Kirchhoff's stress tensor σ^{ij}

$$\text{IVW} = \int_{\Omega} \sigma^{ij} \delta e_{ij} \, d\Omega \tag{20}$$

in which Ω = volume in the undeformed configuration. This expression may be considered evaluated in the following system of convected coordinates $\xi^1\xi^2\xi^3$:

$\xi^1 = s$
ξ^2 = union of all the centerlines of elements at any cross-section
ξ^3 = thickness-wise coordinate

For an elastic material, σ^{ij} may be expressed through the constitutive equations in terms of $e_{11}, e_{12}, \ldots, e_{33}$ so that eqn (21) becomes a functional of e_{ij}. The strain tensor in its turn is expressible, through the kinematical assumptions, in terms of the independent displacement parameters (see [6, 7]), in our case the components of v and the independent components of $\mathbf{f}_\alpha$ $(\alpha = 1, 2, \ldots, n)$.

Up to now, however, only a mean strain tensor is available, from eqns (14) and (15). In order to carry out the volume integration it will be assumed that:

(a) ϵ_{00}, the actual strain, does not deviate from its mean value, $\bar{\epsilon}_{00}$, thickness-wise.

(b) the deviation from the mean of the shearing strains in element α is related only to the change in specific twist of the element. (Note that the theory accommodates beams with a nonzero specific twist in the initial geometry.)

If ω_α denotes the change in specific twist for element α, there are at least two possible ways to account for the deviation prescribed in (b). The first is to assume an expression for $\epsilon_{0\alpha}$ (the actual strain) in terms of the mean strain $\bar{\epsilon}_{0\alpha}$ such as:

$$\epsilon_{0\alpha} = \bar{\epsilon}_{0\alpha} + \omega_\alpha d \tag{21}$$

in which d is a properly chosen thickness-wise coordinate on element α. Note that this could require also a definition of an "equivalent torsional cross-section", in the same spirit as with an "equivalent shearing area".

The second way is to account for the deviations by postulating a contribution to the internal virtual work of the form:

$$\int_0^L T^\alpha \delta\omega_\alpha \, ds \tag{22}$$

in which T^α is a properly chosen function of ω_α and possibly also of $\bar{\epsilon}_{0\alpha}$.

This second way will be adopted here, leaving open the question of the functional dependence of T^α on ω_α and $\bar{\epsilon}_{0\alpha}$. Note however, that for a linearly elastic material a good choice (at least for small ω_α) is:

$$T^\alpha = \frac{1}{3} G l_\alpha h_\alpha^{\ 3} \omega_\alpha \qquad \text{(no sum on } \alpha\text{)} \tag{23}$$

in which G = the shear modulus, l_α = the length of element α, and h_α = the thickness of element α.

Another problem that arises is to express the change in specific twist in terms of $\mathbf{f}_\alpha$ and $\hat{\mathbf{f}}_\alpha$ and their derivatives. Many possibilities may be equally satisfactory, provided they all tend (for small strains) to the projection of $\hat{\mathbf{f}}'_\alpha$ on the plane normal to the deformed axis minus its pre-deformation counterpart. One possible choice for $n \geq 2$, (where n = the number of plate elements in the cross-section) is to express (in the undeformed configuration) the unit vector $\mathbf{n}_\alpha$ normal to $\mathbf{f}_\alpha$ in the plane of the cross-section as some (non-unique if $n > 2$) linear combination of all the $\mathbf{f}_\alpha$. That is, let

$$\mathbf{n}_\alpha = C_\alpha^{\ \beta} \mathbf{f}_\beta. \tag{24}$$

Then the quantity

$$\omega_\alpha = C_\alpha^{\ \beta} [\hat{\mathbf{f}}'_\alpha \cdot \hat{\mathbf{f}}_\beta - \mathbf{f}'_\alpha \cdot \mathbf{f}_\beta] \qquad \text{(no sum on } \alpha\text{)} \tag{25}$$

is an adequate measure of the change in specific twist. Note that because of the constraints (eqns (17) and (18)) it is reasonable to assume that all the elements undergo the same change in specific twist, say ω. Equation (25) could then be evaluated for, say, the first element only. This procedure was adopted in the numerical examples presented below.

Returning to the internal virtual work expression (eqn (20)) and introducing the expressions for the mean strains (eqns (14) and (15)) and the assumption embodied in eqn (22), the following result is obtained:

$$\begin{aligned}
\text{IVW} &= \int_\Omega \sigma^{ij} \delta e_{ij} \, d\Omega \\
&= \sum_{\alpha=1}^{n} \int_{\Omega^\alpha} (\sigma^{11} \delta\bar{\epsilon}_{00} + 2\sigma^{12} \delta\bar{\epsilon}_{0\alpha}) \, d\Omega^\alpha + \int_0^L T^\beta \delta\omega_\beta \, ds \\
&= \int_0^L \{N(\mathbf{r}'_0 + \mathbf{v}') \cdot \delta\mathbf{v}' + M^\beta [(\mathbf{r}'_0 + \mathbf{v}') \cdot \delta\hat{\mathbf{f}}'_\beta + \hat{\mathbf{f}}'_\beta \cdot \delta\mathbf{v}'] \\
&\quad + 2B^{\beta\alpha} \hat{\mathbf{f}}'_\beta \cdot \delta\hat{\mathbf{f}}'_\alpha + S^\alpha [(\mathbf{r}'_0 + \mathbf{v}') \cdot \delta\hat{\mathbf{f}}_\alpha + \hat{\mathbf{f}}_\alpha \cdot \delta\mathbf{v}'] \\
&\quad + Q^{\alpha\beta} (\hat{\mathbf{f}}'_\beta \cdot \delta\hat{\mathbf{f}}_\alpha + \hat{\mathbf{f}}_\alpha \cdot \delta\hat{\mathbf{f}}'_\beta) + T^\beta \delta\omega_\beta\} \, ds
\end{aligned} \tag{26}$$

where the following notation has been used:

$$N = \sum_{\alpha=1}^{n} \int_{A^\alpha} \sigma^{11} \, dA^\alpha \tag{27}$$

$$M^{\beta}=\sum_{\alpha=1}^{n}\int_{A^{\alpha}}\sigma^{11}s^{\beta}\,\mathrm{d}A^{\alpha} \tag{28}$$

$$B^{\beta\rho}=\sum_{\alpha=1}^{n}\int_{A^{\alpha}}\sigma^{11}s^{\beta}s^{\rho}\,\mathrm{d}A^{\alpha} \tag{29}$$

$$S^{\alpha}=\int_{A^{\alpha}}\sigma^{12}\,\mathrm{d}A^{\alpha} \tag{30}$$

$$Q^{\alpha\beta}=\int_{A^{\alpha}}\sigma^{12}s^{\beta}\,\mathrm{d}A^{\alpha} \tag{31}$$

in which A^{α} is the area of the α-th element.

When using eqn (25), the expression for $\delta\omega_{\beta}$ in eqn (26) is

$$\delta\omega_{\beta}=C_{\beta}{}^{\alpha}(\hat{\mathbf{f}}'_{\beta}\cdot\delta\hat{\mathbf{f}}_{\alpha}+\hat{\mathbf{f}}_{\alpha}\cdot\delta\hat{\mathbf{f}}'_{\beta})\qquad(\text{no sum on }\beta). \tag{32}$$

6. EXTERNAL VIRTUAL WORK, EQUILIBRIUM EQUATIONS, BOUNDARY CONDITIONS

It is a straightforward matter to write down the external virtual work expression for a given type of loading. The simplest case is represented by "dead"load, i.e. a load which is independent of the deformation. Then, if $\mathbf{p}(s, s^{\alpha})$ represents the force per unit area of the undeformed beam, the external virtual work (EVW) is given by:

$$\mathrm{EVW}=\int_{A}\mathbf{p}\cdot\delta\hat{\mathbf{r}}\,\mathrm{d}A \tag{33}$$

in which A = the area of the undeformed beam.

Follower loads lead to a similar expression, but with $\mathbf{p}$ depending also on $\hat{\mathbf{r}}$.

Using the principle of virtual work

$$\mathrm{IVW}=\mathrm{EVW} \tag{34}$$

and equating the coefficients of the variations of independent displacement variables a set of equilibrium equations and normal boundary conditions is obtained.

From the numerical standpoint, one can generate a set of discretized equations directly from the principle of virtual work and this was the technique used for solving the numerical examples presented below. To account for the constraints, Lagrange multipliers may be used or the constraints may be used to express some variables in terms of others (see numerical examples).

7. NUMERICAL EXAMPLES

Consider (Fig. 2) the case of an initially straight symmetric "I" beam. Choosing for "O" the center of symmetry of the cross-section the coordinates s^{α} vary as shown in Fig. 3.

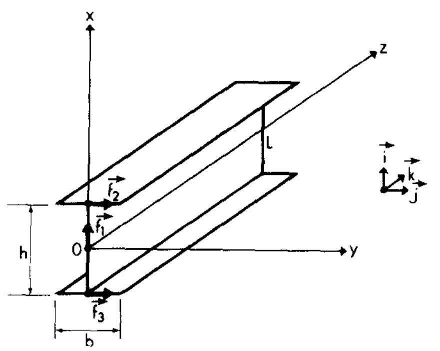

Fig. 2.

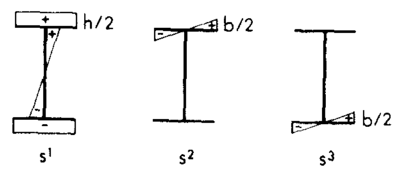

Fig. 3.

Fig. 2. I-Beam geometry.

Fig. 3. s^{α} Coordinates for I section.

For this particular case a measure of change in specific twist for the cross-section is

$$\omega = -\frac{1}{2}\hat{\mathbf{f}}_1' \cdot (\hat{\mathbf{f}}_2 + \hat{\mathbf{f}}_3) \tag{35}$$

(this is a particular case of eqn (24)) and the corresponding contribution to the internal virtual work is

$$\int_0^L T\delta\omega \, \mathrm{d}s \tag{36}$$

with

$$T = \sum_{\alpha=1}^{3} T^\alpha \tag{37}$$

where T^α is given in eqn (23).

To ease the problem of satisfying the constraints, polar coordinates (Fig. 4) are used for expressing the components of $\hat{\mathbf{f}}_\alpha$. Thus:

$$\hat{\mathbf{f}}_1 = \cos\theta_1 \cos\phi_1 \mathbf{i} + \sin\theta_1 \mathbf{j} + \cos\theta_1 \sin\phi_1 \mathbf{k} \tag{38}$$

$$\hat{\mathbf{f}}_\alpha = \sin\theta_\alpha \mathbf{i} + \cos\theta_\alpha \cos\phi_\alpha \mathbf{j} + \cos\theta_\alpha \sin\phi_\alpha \mathbf{k} \qquad \alpha = 2, 3. \tag{39}$$

In this way the three constraints:

$$\hat{\mathbf{f}}_\alpha \cdot \hat{\mathbf{f}}_\alpha = 1 \qquad \alpha = 1, 2, 3 \qquad (\text{no sum on } \alpha) \tag{40}$$

are satisfied identically, i.e. eqn (17) is satisfied.

As for the two perpendicularity constraints arising from eqn (18) they assume the form:

$$\theta_\alpha = \operatorname{atan} \frac{-\tan\theta_1 \cos\phi_\alpha - \sin\phi_1 \sin\phi_\alpha}{\cos\phi_1} \qquad \alpha = 2, 3 \tag{41}$$

where atan is defined between $-\pi/2$ and $\pi/2$. Note that singularities occur for $\theta_\alpha = \pm\pi/2$ and also for $\phi_\alpha = \pm\pi/2$. This implies that $\hat{\mathbf{f}}_1$ may not reach the **j**, **k** plane nor may $\hat{\mathbf{f}}_2$ or $\hat{\mathbf{f}}_3$ reach the **i**, **k** plane.

The independent kinematical unknowns are ϕ_1, θ_1, ϕ_2, ϕ_3 and the three components of **v**. (Equation (41) defines θ_1 and θ_2 in terms of the first four variables).

Discretization is achieved by dividing the beam axis into a number of finite elements and using cubic interpolation for all the unknowns (except ϕ_2 and ϕ_3, for which linear interpolation is used). The discretization technique is similar to that used in [8] and will not be discussed here.

A computer program was developed which solves the system of nonlinear equations by means of the Newton-Raphson algorithm. The program allows for loading and supporting of the beam at any point of its cross-section.

Two numerical examples were selected for which experimental results in the elastic range are available [9] for lateral post-buckling behaviour of "I" beams. In the first example a cantilever is subjected to the action of its own weight and of a varying concentrated force at the tip (acting on the center of the cross-section). The second example is similar, except that a simply supported beam is considered.

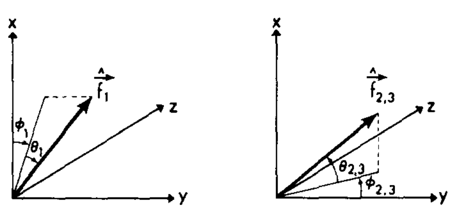

Fig. 4. Polar parameters.

As is seen in Figs. 5–7 the agreement with experiments is very good. The different initial behaviour is due to the effect of imperfections, which in the program were simulated by an initial torque (see Figs. 5 and 7). Geometrical dimensions, element subdivision, elastic constants and loading details are presented in Fig. 8.

8. SUMMARY AND CONCLUSIONS

A theoretical formulation for three-dimensional large deformation analysis of thin walled beams has been presented. The theory evolves from a set of physically reasonable kinematical assumptions and is free of non-objective elements and valid, therefore, for arbitrarily large

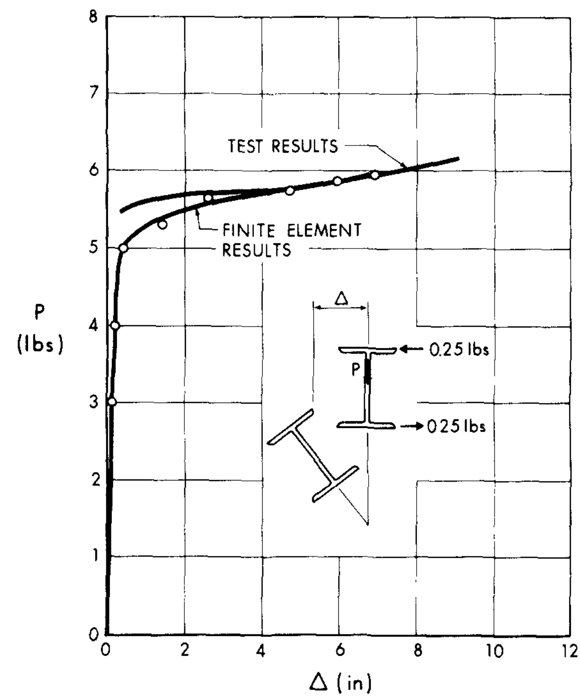

Fig. 5. Cantilever beam—comparative results for horizontal deflection.

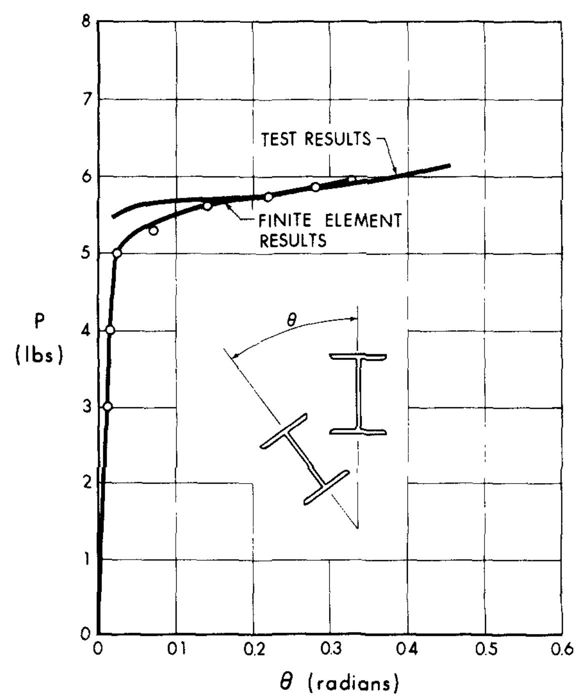

Fig. 6. Cantilever beam—comparative results for twist.

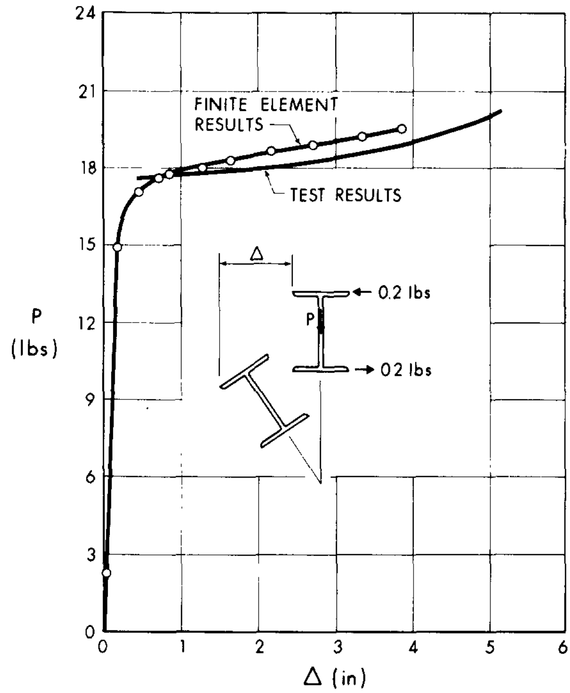

Fig. 7. Simply supported beam—comparative results for horizontal deflection.

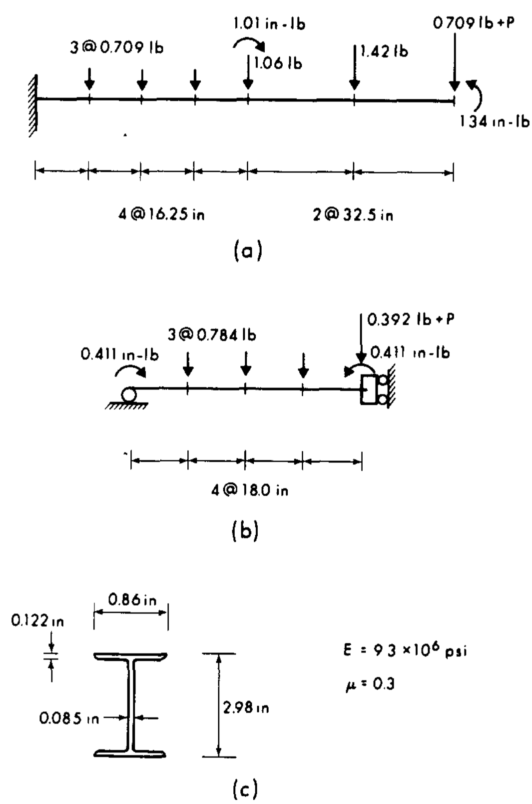

Fig. 8. (a) Geometry and loads for cantilever beam example. (b) Geometry and loads for simply supported beam example. (c) Cross section dimensions.

displacements and rotations. Although the resulting equations and constraints are quite involved they are manageable to the point of being directly programmable for a computer.

The numerical examples show that the model is consistent with experimental results, opening the way for predicting real behaviour of structural elements in the large deformation range. Plastic behaviour could be incorporated in the theory through the constitutive equations.

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